1862

De lineis curvis, quarum rectificatio per datam quadraturam mensuratur

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XXI.

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1. Satis notum est problema inter Geometras olim multum agitatum, quo lineae curvae quaerebantur, quarum rectificatio a datae curvae quadratura pendeat, cujus solutionem etiam Hermannus beatae memoriae contra exspectationem summorum Geometrarum ita feliciter expedivit, ut non solum infinitas curvas algebraicas, quarum rectificatio a data quadratura penderet, exhibuerit, sed etiam hanc conditionem adjunxerit, ut istae curvae unum duosve atque adeo tot, quot lubuerit, haberent arcus absolute rectificabiles. Cum autem methodus, qua Hermannus erat usus, nimis videretur recondita, neque ad uberiorem usum in Analysi satis accommodata, aliam methodum planam ac facilem investigavi, cujus ope non solum hoc problema, sed etiam omnia, quae hujus generis occurrere queant, expedite resolvi possunt. Complectitur ista methodus quasi novam Analyseos speciem, cujus elementa, quae multo latius patere videntur, dilucide exposui in Novis Commentariis Academiae imperialis Petropolitanae.

2. Hujus methodi beneficio, si proponatur quadratura seu formula integralis quaecunque $\int Zdz$, existente Z functione ipsius z quacunque, innumerabiles curvae algebraicae definiri possunt, quarum rectificatio ab ista formula ita pendeat, ut ejus integratione concessa omnes harum curvarum arcus indefinite definiri queant. Per variabilem scilicet z ejusmodi expressiones algebraicae pro coordinatis x et y assignantur, ut inde formulae $\sqrt{(dx^2+dy^2)}$ integratio perducatur ad hujusmodi formam $\alpha\int Zdz+V$, ubi V sit functio algebraica ipsius z. Verum haec quantitas V non arbitrio nostro relinquitur, etiamsi infinitis modis variari queat: atque hinc ope methodi a me traditae problema non ita resolvi potest, ut curvarum inveniendarum arcus absolute per formulam propositam $\int Zdz$ ejusve multiplum $\alpha\int Zdz$ exprimantur.

3. Maxime igitur diversum est problema, quo quaeruntur curvae algebraicae, quarum arcus per propositam quampiam formulam integralem $\int Zdz$ simpliciter, sine adjunctione cujusdam functionis algebraicae exprimantur. Atque adeo hoc problema saepenumero ne solutionem quidem admittere videtur. Ita si sit $Z=\frac{a}{z}$, et curva algebraica sit investiganda, cujus arcus per $a\int\frac{dz}{z}$, seu alz exprimatur, vehementer dubito, num quisquam unquam hujusmodi curvam sit reperturus?

Quaestio scilicet huc redit, ut ejusmodi binae functiones algebraicae ipsius z inveniantur, quae pro coordinatis x et y substitutae praebeant $\sqrt{(dx^2+dy^2)} = \frac{adz}{z}$. Postquam equidem hoc problema multis modis tentavi, aliisque insignibus Geometris enodandum proposui, neque ego, neque quisquam alius solutionem assequi potuimus: cum tamen in genere si quaeratur curva algebraica, cujus rectificatio a logarithmis pendeat, problema sit facillimum, atque adeo parabola conica ei satisfaciat. Unde concludendum est hoc problema vel omnino nullam solutionem admittere, vel methodum adhuc plane nobis incognitam requirere.

4. Evenire quoque posse videtur, ut hujusmodi problemata unicam tantum solutionem admittant, neque plus una curva exhiberi queat, cujus arcus per datam formulam integralem exprimantur. Equidem hoc sum expertus in formula $\int \frac{adz}{\sqrt{(aa-zz)}}$, qua arcus circuli exprimitur: nullam enim aliam lineam curvam algebraicam invenire potui, cujus arcus per eandem formulam exprimeretur. Sic nulla videtur extare curva algebraica, cujus arcui cuicunque aequalis arcus circularis exhiberi queat, etiamsi innumerabiles lineae algebraicae sint notae, quarum rectificatio a rectificatione circuli pendeat. Statim enim atque hae curvae a circulo sunt diversae, earum arcus aequantur aggregato ex arcu quodam circulari et linea geometrice assignabili, quae nonnisi certis casibus in nihilum abire potest. Idem tenendum est de formulis $\int \frac{adz}{\sqrt{(2az-zz)}}$ et $\int \frac{aadz}{aa+zz}$ aliisque, in quas illa formula $\int \frac{adz}{\sqrt{(aa-zz)}}$ per substitutiones transformari potest.

5. Dantur tamen etiam ejusmodi formulae $\int Zdz$, pro quibus innumerabiles curvae algebraicae exhiberi possunt, ita ut infinitae curvae algebraicae assignari queant, in quarum una si capiatur arcus quicunque, in reliquis omnibus pares arcus abscindere liceat. Huc imprimis pertinet problema olim a Celebb. Bernoulliis tractatum, quo curva algebraica quaerebatur, cujus rectificatio cum rectificatione curvae elasticae conveniret, seu per hanc formulam $\int \frac{aadz}{\sqrt{(a^4-z^4)}}$ exprimeretur: invenerunt enim lineam quarti ordinis, ob figuram *lemniscatam* dictam, quae huic scopo satisfaceret. Ostendam autem praeter lemniscatam infinitas alias exhiberi posse curvas algebraicas, quarum arcus generatim per eandem formulam exprimantur. Cum igitur lemniscata, docente Ill. Fagnano, hanc habeat insignem proprietatem, ut in ea perinde atque in circulo, arcus quotcunque aequales abscindi queant, eadem proprietas quoque in omnes curvas, quarum arcus per eandem formulam $\int \frac{aadz}{\sqrt{(a^4-z^4)}}$ exprimuntur, competet; quae ergo merentur, ut diligentius evolvantur.

6. Methodus quidem, qua hanc investigationem suscipio, per se satis est plana, et ope calculi angulorum facile expediri potest. Si enim arcus cujuspiam curvae per hanc formulam $\int Zdz$ debeat exprimi, vocatis coordinatis orthogonalibus x et y, atque introducto angulo quocunque φ, statuatur

$$dx = Zdz \cos\varphi \quad \text{et} \quad dy = Zdz \sin\varphi$$

sic enim prodibit arcus elementum

$$\sqrt{(dx^2+dy^2)} = Zdz, \text{ ipseque arcus } = \int Zdz.$$

Unde quaestio huc redit, ut quemadmodum arcus φ ad variabilem z comparatus esse debeat, investigetur, ut ambae formulae $Zdz \cos\varphi$ et $Zdz \sin\varphi$ evadant integrabiles: quippe quod conditio qua curvae debent esse algebraicae, postulat. Hunc in finem illae integrationes per solos sinus et cosinus angulorum sunt absolvendae, neque ipsi anguli, qui formulas redderent transcendentes, sunt admittendi.

De curva lemniscata.

7. Propositum ergo sit curvas algebraicas investigare, quarum arcus indefinite per hanc formulam integralem $\int\frac{aadz}{\sqrt{(a^4-z^4)}}$ exprimantur, et positis coordinatis orthogonalibus x et y statuamus

$$dx=\frac{aadz}{\sqrt{(a^4-z^4)}}\cos\varphi \quad \text{et} \quad dy=\frac{aadz}{\sqrt{(a^4-z^4)}}\sin\varphi,$$

quas formulas absolute integrabiles reddi oportet. Ut partem $\frac{aadz}{\sqrt{(a^4-z^4)}}$ quoque ad calculum angulorum perducam, pono $zz=aa\sin\theta$, ut fiat $\sqrt{(a^4-z^4)}=aa\cos\theta$, et ob $z=a\sqrt{\sin\theta}$ erit

$$dz=\frac{ad\theta\cos\theta}{2\sqrt{\sin\theta}} \quad \text{et} \quad \frac{aadz}{\sqrt{(a^4-z^4)}}=\frac{ad\theta}{2\sqrt{\sin\theta}}.$$

Hinc itaque nostrae formulae integrabiles reddendae sunt

$$dx=\frac{ad\theta\cos\varphi}{2\sqrt{\sin\theta}} \quad \text{et} \quad dy=\frac{ad\theta\sin\varphi}{2\sqrt{\sin\theta}}.$$

Ponamus ergo $\varphi=n\theta$, ut sit

$$\frac{2dx}{a}=\frac{d\theta\cos n\theta}{\sqrt{\sin\theta}} \quad \text{et} \quad \frac{2dy}{a}=\frac{d\theta\sin n\theta}{\sqrt{\sin\theta}},$$

et videamus quinam valores pro n sumti has ambas formulas integrabiles reddant.

8. Consideremus in genere has formulas

$$\frac{d\theta\cos m\theta}{\sqrt{\sin\theta}} \quad \text{et} \quad \frac{d\theta\sin m\theta}{\sqrt{\sin\theta}}$$

et perpendamus quomodo ad simpliciores revocari possunt. Talis enim reductio unica via esse videtur ad casus integrabilitatis eruendos. Statuamus ergo primo

$$P=\cos(m-1)\theta.\sqrt{\sin\theta},$$

et differentiando habebitur

$$dP=\frac{-(m-1)d\theta\sin(m-1)\theta.\sin\theta+\frac{1}{2}d\theta\cos(m-1)\theta.\cos\theta}{\sqrt{\sin\theta}}.$$

Cum autem sit
$$\sin\alpha\theta\sin\theta=\frac{1}{2}\cos(\alpha-1)\theta-\frac{1}{2}\cos(\alpha+1)\theta$$

et
$$\cos\alpha\theta\cos\theta=\frac{1}{2}\cos(\alpha-1)\theta+\frac{1}{2}\cos(\alpha+1)\theta,$$

erit
$$dP=\frac{-(2m-3)d\theta\cos(m-2)\theta+(2m-1)d\theta\cos m\theta}{4\sqrt{\sin\theta}},$$

unde obtinetur

$$\int\frac{d\theta\cos m\theta}{\sqrt{\sin\theta}}=\frac{4\cos(m-1)\theta\sqrt{\sin\theta}}{2m-1}+\frac{2m-3}{2m-1}\int\frac{d\theta\cos(m-2)\theta}{\sqrt{\sin\theta}}.$$

9. Si deinde simili modo statuamus

$$Q=\sin(m-1)\theta\sqrt{\sin\theta},$$

erit differentiando

$$dQ = \frac{(m-1)\,d\theta \cos(m-1)\theta \sin\theta + \frac{1}{2}\,d\theta \sin(m-1)\theta \cos\theta}{\sqrt{\sin\theta}}.$$

Cum vero sit

$$\cos\alpha\theta\sin\theta = -\frac{1}{2}\sin(\alpha-1)\theta + \frac{1}{2}\sin(\alpha+1)\theta$$

$$\text{et}\quad \sin\alpha\theta\cos\theta = +\frac{1}{2}\sin(\alpha-1)\theta + \frac{1}{2}\sin(\alpha+1)\theta,$$

erit per has substitutiones

$$dQ = \frac{-(2m-3)\,d\theta\sin(m-2)\theta + (2m-1)\,d\theta\sin m\theta}{4\sqrt{\sin\theta}}.$$

Unde singulis partibus integratis consequemur

$$\int\frac{d\theta\sin m\theta}{\sqrt{\sin\theta}} = \frac{4\sin(m-1)\theta\sqrt{\sin\theta}}{2m-1} + \frac{2m-3}{2m-1}\int\frac{d\theta\sin(m-2)\theta}{\sqrt{\sin\theta}},$$

hincque ergo patet, si formulae propositae $\frac{d\theta\cos n\theta}{\sqrt{\sin\theta}}$ et $\frac{d\theta\sin n\theta}{\sqrt{\sin\theta}}$ fuerint integrabiles casu $n=\lambda$, tum etiam integrabiles esse futuras casibus

$$n=\lambda+2,\quad n=\lambda+4,\quad n=\lambda+6,\ \text{etc.},$$

sicque ex uno infinitos resultare casus integrabiles.

10. Ex his autem reductionibus statim unus se offert casus absolute integrabilis, scilicet quando $2m-3=0$ seu $m=\frac{3}{2}$; unde obtinemus

$$\int\frac{d\theta}{\sqrt{\sin\theta}}\cos\frac{3}{2}\theta = 2\cos\frac{1}{2}\theta\sqrt{\sin\theta}\quad\text{et}\quad\int\frac{d\theta}{\sqrt{\sin\theta}}\sin\frac{3}{2}\theta = 2\sin\frac{1}{2}\theta\sqrt{\sin\theta}.$$

Deinde integratio succedet casu $m=\frac{7}{2}$ seu $2m=7$, unde fit

$$\int\frac{d\theta}{\sqrt{\sin\theta}}\cos\frac{7}{2}\theta = \frac{2}{3}\cos\frac{5}{2}\theta\sqrt{\sin\theta} + 2\,.\,\frac{2}{3}\cos\frac{1}{2}\theta\sqrt{\sin\theta},$$

$$\int\frac{d\theta}{\sqrt{\sin\theta}}\sin\frac{7}{2}\theta = \frac{2}{3}\sin\frac{5}{2}\theta\sqrt{\sin\theta} + 2\,.\,\frac{2}{3}\sin\frac{1}{2}\theta\sqrt{\sin\theta}.$$

Hinc progressus patet ad casum $m=\frac{11}{2}$ seu $2m=11$, qui dat

$$\int\frac{d\theta}{\sqrt{\sin\theta}}\cos\frac{11}{2}\theta = \frac{2}{5}\cos\frac{9}{2}\theta\sqrt{\sin\theta} + \frac{2\,.\,4}{3\,.\,5}\cos\frac{5}{2}\theta\sqrt{\sin\theta} + 2\,.\,\frac{2\,.\,4}{3\,.\,5}\cos\frac{1}{2}\theta\sqrt{\sin\theta},$$

$$\int\frac{d\theta}{\sqrt{\sin\theta}}\sin\frac{11}{2}\theta = \frac{2}{5}\sin\frac{9}{2}\theta\sqrt{\sin\theta} + \frac{2\,.\,4}{3\,.\,5}\sin\frac{5}{2}\theta\sqrt{\sin\theta} + 2\,.\,\frac{2\,.\,4}{3\,.\,5}\sin\frac{1}{2}\theta\sqrt{\sin\theta},$$

et sequens casus $m=\frac{15}{2}$ praebebit

$$\int\frac{d\theta}{\sqrt{\sin\theta}}\cos\frac{15}{2}\theta = \left(\frac{2}{7}\cos\frac{13}{2}\theta + \frac{2\,.\,6}{5\,.\,7}\cos\frac{9}{2}\theta + \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\cos\frac{5}{2}\theta + 2\,.\,\frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\cos\frac{1}{2}\theta\right)\sqrt{\sin\theta},$$

$$\int\frac{d\theta}{\sqrt{\sin\theta}}\sin\frac{15}{2}\theta = \left(\frac{2}{7}\sin\frac{13}{2}\theta + \frac{2\,.\,6}{5\,.\,7}\sin\frac{9}{2}\theta + \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\sin\frac{5}{2}\theta + 2\,.\,\frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\sin\frac{1}{2}\theta\right)\sqrt{\sin\theta}.$$

11. Ut in coëfficientibus angulorum fractiones evitemus, ponamus $\theta = 2\omega$, ut sit

$$zz = aa\sin 2\omega, \text{ seu } \sin 2\omega = \frac{zz}{aa},$$

unde erit

$$\sin\omega = \frac{1}{2}\sqrt{\left(1 + \frac{zz}{aa}\right)} - \frac{1}{2}\sqrt{\left(1 - \frac{zz}{aa}\right)} \quad \text{et} \quad \cos\omega = \frac{1}{2}\sqrt{\left(1 + \frac{zz}{aa}\right)} + \frac{1}{2}\sqrt{\left(1 - \frac{zz}{aa}\right)}.$$

Atque infinitas curvas algebraicas exhibere poterimus, quarum arcus seu valor integralis

$$\int\sqrt{(dx^2 + dy^2)}$$

praecise fiat aequalis formulae

$$\int\frac{aadz}{\sqrt{(a^4 - z^4)}} = a\int\frac{d\omega}{\sqrt{\sin 2\omega}}.$$

Ac curva quidem prima eaque simplicissima his continebitur coordinatis

$$x = a\cos\omega\sqrt{\sin 2\omega} \quad \text{et} \quad y = a\sin\omega\sqrt{\sin 2\omega},$$

ex quibus fit $xx + yy = aa\sin 2\omega$ et $\sqrt{(xx + yy)} = a\sqrt{\sin 2\omega}$. Hinc ergo porro elicitur

$$\cos\omega = \frac{x}{\sqrt{(xx + yy)}} \quad \text{et} \quad \sin\omega = \frac{y}{\sqrt{(xx + yy)}},$$

ideoque $\sin 2\omega = 2\sin\omega\cos\omega = \frac{2xy}{xx + yy}$. Quo valore substituto habebitur aequatio inter solas x et y pro curva

$$(xx + yy)^2 = 2aaxy,$$

quae est ipsa aequatio lemniscatae.

12. Secunda curva algebraica, cujus arcus per eandem formulam

$$\int\frac{aadz}{\sqrt{(a^4 - z^4)}} = a\int\frac{d\omega}{\sqrt{\sin 2\omega}},$$

exprimuntur, continebitur his coordinatis

$$x = \frac{a}{3}(\cos 5\omega + 2\cos\omega)\sqrt{\sin 2\omega},$$

$$y = \frac{a}{3}(\sin 5\omega + 2\sin\omega)\sqrt{\sin 2\omega}.$$

Tertia porro curva aeque satisfaciens his:

$$x = \frac{a}{5}\left(\cos 9\omega + \frac{4}{3}\cos 5\omega + \frac{4.2}{3.1}\cos\omega\right)\sqrt{\sin 2\omega},$$

$$y = \frac{a}{5}\left(\sin 9\omega + \frac{4}{3}\sin 5\omega + \frac{4.2}{3.1}\sin\omega\right)\sqrt{\sin 2\omega}.$$

Quarta vero his:

$$x = \frac{a}{7}\left(\cos 13\omega + \frac{6}{5}\cos 9\omega + \frac{6.4}{5.3}\cos 5\omega + \frac{6.4.2}{5.3.1}\cos\omega\right)\sqrt{\sin 2\omega},$$

$$y = \frac{a}{7}\left(\sin 13\omega + \frac{6}{5}\sin 9\omega + \frac{6.4}{5.3}\sin 5\omega + \frac{6.4.2}{5.3.1}\sin\omega\right)\sqrt{\sin 2\omega}.$$

Quinta hinc sponte formari potest.

$$x = \frac{a}{9}\left(\cos 17\omega + \frac{8}{7}\cos 13\omega + \frac{8.6}{7.5}\cos 9\omega + \frac{8.6.4}{7.5.3}\cos 5\omega + \frac{8.6.4.2}{7.5.3.1}\cos\omega\right)\sqrt{\sin 2\omega},$$

$$y = \frac{a}{9}\left(\sin 17\omega + \frac{8}{7}\sin 13\omega + \frac{8.6}{7.5}\sin 9\omega + \frac{8.6.4}{7.5.3}\sin 5\omega + \frac{8.6.4.2}{7.5.3.1}\sin\omega\right)\sqrt{\sin 2\omega}.$$

13. Sic igitur infinitas nacti sumus curvas algebraicas, quarum rectificatio plane congruit cum rectificatione lemniscatae, ita ut cuique arcui hujus curvae in omnibus illis arcus aequales abscindi possint; vix tamen asseverare ausim, praeter has nullas dari alias curvas algebraicas, quae eadem praeditae sint proprietate. Methodus enim, qua sum usus, non ita est comparata, ut pro generali haberi possit, propterea quod in formulis § 7 angulum φ tanquam multiplum anguli θ spectavi, cum tamen fortasse alia relatio inter eos intercedere possit, quae ad integrationem aeque sit accomodata. Hoc inde suspicari licet, quod si aliae formulae integrales $\int Z dz$ proponantur, eaeque pari modo ad angulum quempiam θ reducantur, integratio non succedat pro angulo φ multiplum anguli θ assumendo, cum tamen saepenumero aliae relationes negotium conficiant. Hujusmodi casus probe notasse juvabit, quoniam inde forte methodum latius patentem talia problemata tractandi derivare licebit, si cunctae operationes, quas varia problemata singularia requirunt, diligenter perpendantur, atque inter se conferantur. Quem in finem unam atque alteram solutionem similium quaestionum adjungam.

De Parabola.

14. Propositum itaque sit alias curvas algebraicas investigare, quarum rectificatio conveniat cum rectificatione parabolae, seu quarum arcus indefinite exprimatur per hanc formulam:

$$\int \frac{dz}{a}\sqrt{(aa + zz)}.$$

Necesse igitur est, ut coordinatae orthogonales ita se habeant

$$x = \int \frac{dz \cos\varphi}{a}\sqrt{(aa + zz)} \quad \text{et} \quad y = \int \frac{dz \sin\varphi}{a}\sqrt{(aa + zz)},$$

ubi definiendum erit, qualem relationem angulus φ ad variabilem z tenere debeat, ut ambae istae formulae integrabiles reddantur. Ponamus ergo $\frac{z}{a} = \operatorname{tang}\theta$, ut fiat $\sqrt{(aa + zz)} = a\sec\theta = \frac{a}{\cos\theta}$, et cum sit $\frac{dz}{a} = \frac{d\theta}{\cos^2\theta}$, erit arcus $\int \frac{dz}{a}\sqrt{(aa + zz)} = \int \frac{a\,d\theta}{\cos^3\theta}$, et coordinatae:

$$x = a\int \frac{d\theta\cos\varphi}{\cos^3\theta} \quad \text{et} \quad y = a\int \frac{d\theta\sin\varphi}{\cos^3\theta},$$

atque hic iterum observo, certa multipla anguli θ pro angulo φ exhiberi posse, quibus ambae formulae integrabiles evadant. Statuatur ergo $\varphi = n\theta$, ut habeamus pro coordinatis sequentes expressiones:

$$x = a\int \frac{d\theta\cos n\theta}{\cos^3\theta} \quad \text{et} \quad y = a\int \frac{d\theta\sin n\theta}{\cos^3\theta}.$$

15. Jam per reductionem formularum integralium, quali supra sum usus, reperiemus

$$\int \frac{d\theta\cos n\theta}{\cos^3\theta} = \frac{2\sin(n-1)\theta}{(n-3)\cos^2\theta} - \frac{(n+1)}{n-3}\int \frac{d\theta\cos(n-2)\theta}{\cos^3\theta}, \quad \int \frac{d\theta\sin n\theta}{\cos^3\theta} = \frac{-2\cos(n-1)\theta}{(n-3)\cos^2\theta} - \frac{(n+1)}{n-3}\int \frac{d\theta\sin(n-2)\theta}{\cos^3\theta}$$

unde patet, si integratio succedat casu quocunque $n=\lambda$, eam quoque succedere casibus

$$n=\lambda+2,\quad n=\lambda+4,\quad n=\lambda+6, \text{ etc.},$$

sicque infinitas curvas algebraicas ex unica impetrari. Patet autem si sit $n=3$, fore

$$\int\frac{d\theta\cos\theta}{\cos^3\theta}=\frac{\sin 2\theta}{2\cos^2\theta}\quad\text{et}\quad\int\frac{d\theta\sin\theta}{\cos^3\theta}=-\frac{\cos 2\theta}{2\cos^2\theta},\quad\text{sive}$$

$$\int\frac{d\theta\cos\theta}{\cos^3\theta}=\frac{\sin\theta}{\cos\theta}\quad\text{et}\quad\int\frac{d\theta\sin\theta}{\cos^3\theta}=+\frac{1}{2\cos^2\theta},$$

quo casu prodit

$$x=\frac{a\sin\theta}{\cos\theta}\quad\text{et}\quad y=\frac{a}{2\cos^2\theta},\quad\text{ergo}\quad\frac{xx}{2a}=\frac{a\sin^2\theta}{2\cos^2\theta},$$

hincque $y-\frac{xx}{2a}=\frac{a}{2}$, quae est aequatio pro ipsa parabola.

16. Verum etiamsi hic unum casum integrabilitatis, quo $\varphi=\theta$ seu $n=1$ habeamus cognitum, tamen singulari fato ex eo nulli alii casus elici possunt. Si enim statuamus $n=3$, ob denominatorem $n-3$ evanescentem, integralia inde pro casu $\varphi=3\theta$ minime reperiuntur. Casu autem $n=-1$ formulae praecedentes redeunt, ita ut propter hoc incommodum nullus aditus ad curvas magis compositas pateat. Videri ergo posset parabola pari conditione praedita ac circulus, ut praeter se ipsam nullas alias agnoscat curvas algebraicas secum commensurabiles. Ex ipsa verum angulorum compositione manifestum est, quicunque numerus integer excepta unitate pro n statuatur, formulam $\int\frac{d\theta\cos n\theta}{\cos^3\theta}$ nunquam integrabilem evadere, sed semper per integrationem ipsum angulum θ induci. Interim tamen alia methodo quaesito satisfieri potest, unde non difficulter talis curva eruitur

$$x=\tfrac{1}{2}z\sqrt{(4+zz)}\ \text{et}\ y=\sqrt{(4+zz)},\ \text{seu}\ y^4=4(xx+yy),\ \text{pro qua est}\ \sqrt{(dx^2+dy^2)}=dz\sqrt{(1+zz)}.$$

De Ellipsi.

17. Progredior ergo ad curvas algebraicas indagandas, quarum arcus cum arcubus ellipseos sint commensurabiles. Quaestio igitur huc redit, ut curvarum inveniendarum arcus exprimantur per hanc formulam $\int dz\sqrt{(1+\frac{mmzz}{1-zz})}$, quae est formula pro arcu elliptico abscissae z respondente, dum applicata est $=m\sqrt{(1-zz)}$. Pro curvis ergo, quas quaerimus, statuamus coordinatas

$$x=\int dz\cos\varphi\sqrt{(1+\frac{mmzz}{1-zz})}\quad\text{et}\quad y=\int dz\sin\varphi\sqrt{(1+\frac{mmzz}{1-zz})},$$

et videamus quomodo angulus φ capi debeat, ut ambae istae formulae fiant integrabiles. Ponamus $z=\sin\theta$, et hae formulae erunt

$$x=\int d\theta\cos\varphi\sqrt{(\cos^2\theta+mm\sin^2\theta)}\quad\text{et}\quad y=\int d\theta\sin\varphi\sqrt{(\cos^2\theta+mm\sin^2\theta)},$$

ubi manifestum est, quaecunque multipla anguli θ pro φ statuantur, has expressiones nullo modo ad integrationem perduci posse. Aliis ergo artificiis erit utendum, siquidem certum est dari curvas algebraicas quaesito satisfacientes.

18. Quoniam irrationalitas negotium turbat, ad ejus speciem saltem tollendam pono

$$m \tang \theta = \tang \omega, \quad \text{ut sit} \quad mm \sin^2 \theta = \cos^2 \theta \tang^2 \omega,$$

$$\text{hincque} \quad \sqrt{(\cos^2 \theta + mm \sin^2 \theta)} = \cos \theta \sqrt{(1 + \tang^2 \omega)} = \frac{\cos \theta}{\cos \omega}.$$

Hac substitutione facta nostrae coordinatae erunt

$$x = \int \frac{d\theta \cos \theta \cos \varphi}{\cos \omega} \quad \text{et} \quad y = \int \frac{d\theta \cos \theta \sin \varphi}{\cos \omega},$$

ubi notandum est angulos θ et ω ita a se invicem pendere, ut sit

$$m \tang \theta = \tang \omega, \quad \text{ideoque} \quad \frac{m\, d\theta}{\cos^2 \theta} = \frac{d\omega}{\cos^2 \omega}.$$

Statuatur jam

$\varphi = n\theta - \omega$, et ob $\cos \varphi = \cos n\theta \cos \omega + \sin n\theta \sin \omega$ et $\sin \varphi = \sin n\theta \cos \omega - \cos n\theta \sin \omega$ coordinatae ita exprimentur, ut sit ob $\tang \omega = m \tang \theta$

$$x = \int d\theta \cos \theta \cos n\theta + \int d\theta \cos \theta \sin n\theta \tang \omega = \int d\theta (\cos \theta \cos n\theta + m \sin \theta \sin n\theta),$$

$$y = \int d\theta \cos \theta \sin n\theta - \int d\theta \cos \theta \cos n\theta \tang \omega = \int d\theta (\cos \theta \sin n\theta - m \sin \theta \cos n\theta),$$

quas formulas, quicunque numerus pro n assumatur praeter unitatem, manifestum est semper esse integrabiles.

19. Cum igitur sit $\cos \theta \cos n\theta = \frac{1}{2} \cos (n-1)\theta + \frac{1}{2} \cos (n+1)\theta$,

$$\sin \theta \sin n\theta = \frac{1}{2} \cos (n-1)\theta - \frac{1}{2} \cos (n+1)\theta,$$

$$\cos \theta \sin n\theta = \frac{1}{2} \sin (n-1)\theta + \frac{1}{2} \sin (n+1)\theta,$$

$$-\sin \theta \cos n\theta = \frac{1}{2} \sin (n-1)\theta - \frac{1}{2} \sin (n+1)\theta,$$

substituendis his valoribus habebimus

$$x = \frac{1}{2} \int d\theta \, ((m+1) \cos (n-1)\theta - (m-1) \cos (n+1)\theta),$$

$$y = \frac{1}{2} \int d\theta \, ((m+1) \sin (n-1)\theta - (m-1) \sin (n+1)\theta),$$

unde valores integrales sponte fluunt:

$$x = \frac{+(m+1) \sin (n-1)\theta}{2(n-1)} - \frac{(m-1) \sin (n+1)\theta}{2(n+1)},$$

$$y = \frac{-(m+1) \cos (n-1)\theta}{2(n-1)} + \frac{(m-1) \cos (n+1)\theta}{2(n+1)}.$$

Hincque cum pro n numeros quoscunque rationales praeter unitatem accipere liceat, innumerabiles lineae algebraicae exhiberi possunt.

20. Cum igitur unitas pro n substitui nequeat, casus simplicissimus prodibit, si ponatur $n = 0$, quo ergo habebitur

$$x = \frac{1}{2} (m+1) \sin \theta - \frac{1}{2} (m-1) \sin \theta = \sin \theta,$$

$$y = \frac{1}{2} (m+1) \cos \theta + \frac{1}{2} (m-1) \cos \theta = m \cos \theta,$$

unde fit $mmxx + yy = mm$, ideoque $y = m\sqrt{(1 - xx)}$, quae est aequatio pro ellipsi proposita, cujus arcus ob $x = \sin\theta = z$ utique est $\int dz\sqrt{(1 + \frac{mmzz}{1-zz})}$, uti requiritur, erit enim

$$x = z \quad \text{et} \quad y = m\sqrt{(1 - zz)}.$$

Aliae vero curvae, quarum eadem est rectificatio, prodibunt, si numero n praeter unitatem alii valores tribuantur. Sit igitur $n = 2$, atque habebitur

$$x = \frac{1}{2}(m+1)\sin\theta - \frac{1}{6}(m-1)\sin 3\theta, \qquad y = -\frac{1}{2}(m+1)\cos\theta + \frac{1}{6}(m-1)\cos 3\theta,$$

unde fit $xx + yy = \frac{1}{4}(m+1)^2 + \frac{1}{36}(m-1)^2 - \frac{1}{6}(mm-1)\cos 2\theta$, seu

$$xx + yy = \frac{5}{18}mm + \frac{4}{9}m + \frac{5}{18} - \frac{1}{6}(mm-1)\cos 2\theta.$$

Verum praestat uti formulis illis pro x et y inventis, quia ad cognoscendam et construendam curvam sunt maxime idoneae.

21. Antequam in evolutione horum casuum ulterius progrediar, notari conveniet, quantitatem m tam negative quam affirmative capi posse, propterea quod in expressione arcus quadratum mm tantum inest. Verumtamen iidem casus resultant, si numerus n negative capiatur, ita ut quantitate m ambigua assumta, non opus sit pro n valores negativos statuere. Hinc ergo quilibet numerus positivus pro n sumtus duas praebet lineas algebraicas, prouti m vel affirmative accipitur, vel negative; sicque post ellipsin has duas habebimus curvas satisfacientes

$$x = \frac{1}{2}(m+1)\sin\theta - \frac{1}{6}(m-1)\sin 3\theta, \qquad x = \frac{1}{2}(m-1)\sin\theta - \frac{1}{6}(m+1)\sin 3\theta,$$

$$y = \frac{1}{2}(m+1)\cos\theta - \frac{1}{6}(m-1)\cos 3\theta, \qquad y = \frac{1}{2}(m-1)\cos\theta - \frac{1}{6}(m+1)\cos 3\theta,$$

ubi quidem valorem ipsius y negative sumsi. Similes fere expressiones prodeunt, si ponatur $n = \frac{1}{2}$, unde quoque hae duae curvae oriuntur

$$x = (m+1)\sin\frac{1}{2}\theta - \frac{1}{3}(m-1)\sin\frac{3}{2}\theta, \qquad x = (m-1)\sin\frac{1}{2}\theta - \frac{1}{3}(m+1)\sin\frac{3}{2}\theta,$$

$$y = (m+1)\cos\frac{1}{2}\theta + \frac{1}{3}(m-1)\cos\frac{3}{2}\theta, \qquad y = (m-1)\cos\frac{1}{2}\theta + \frac{1}{3}(m+1)\cos\frac{3}{2}\theta.$$

Atque evidens est eliminando arcu θ has quatuor aequationes ad eundem ordinem esse ascensuras.

22. Ponamus $n = 3$, hincque duae nascentur curvae istae

$$x = \frac{1}{4}(m+1)\sin 2\theta - \frac{1}{8}(m-1)\sin 4\theta, \qquad x = \frac{1}{4}(m-1)\sin 2\theta - \frac{1}{8}(m+1)\sin 4\theta,$$

$$y = \frac{1}{4}(m+1)\cos 2\theta - \frac{1}{8}(m-1)\cos 4\theta, \qquad y = \frac{1}{4}(m-1)\cos 2\theta - \frac{1}{8}(m+1)\sin 4\theta,$$

At si ponamus $n = \frac{1}{3}$, non multum absimiles hae curvae nascuntur

$$x = \frac{3}{4}(m+1)\sin\frac{2}{3}\theta - \frac{3}{8}(m-1)\sin\frac{4}{3}\theta, \qquad x = \frac{3}{4}(m-1)\sin\frac{2}{3}\theta - \frac{3}{8}(m+1)\sin\frac{4}{3}\theta,$$

$$y = \frac{3}{4}(m+1)\cos\frac{2}{3}\theta + \frac{3}{8}(m-1)\cos\frac{4}{3}\theta, \qquad y = \frac{3}{4}(m-1)\cos\frac{2}{3}\theta + \frac{3}{8}(m+1)\cos\frac{4}{3}\theta.$$

Omnes enim hae quatuor curvae tantum ad ordinem linearum quartum referuntur. Ex quibus perspicuum est, quomodo ex quavis hypothesi quaternae curvae elici queant, ad eundem ordinem referendae, nisi quatenus forte casu ordo deprimi possit. Haec ergo infinita linearum algebraicarum multitudo, quarum arcus omnes per arcus ellipticos absolute mensurantur, omnino est notatu digna, idque eo magis, quod pro omnibus coordinatae x et y binis tantum terminis exprimuntur; unde earum constructio haud parum concinna adornari potest, etiamsi plerumque curvae ad altiores linearum ordines referantur.

23. De his autem omnibus lineis imprimis est notandum, eas ad classem Epicycloidum et Hypocycloidum pertinere ac per motum volutorium circuli super peripheria alterius circuli, sive extus sive intus describi posse. Hoc autem hae curvae a vulgaribus epicycloidibus et hypocycloidibus differunt, quod in circulo mobili punctum describens non in ejus peripheria, sed sive extra sive intra eam assumi debet. Si enim in peripheria caperetur, quo casu epicycloides et hypocycloides vulgares prodirent, curvae descriptae absolute essent rectificabiles, neque idcirco ad nostrum institutum essent accommodatae: sin autem punctum describens in ipso centro circuli mobilis assumeretur, curva descripta perpetuo foret circulus. Verum sive punctum describens capiatur extra, sive intra peripheriam circuli mobilis, hoc modo semper curvae describuntur, quarum rectificatio per arcus ellipticos absolute confici potest. Nostrae ergo curvae prodibunt, si distantia puncti describentis a centro circuli mobilis sive major fuerit sive minor quam ejus semidiameter.

24. Natura autem hujusmodi linearum accuratius perpensa, curvae, quarum arcus per arcus datae ellipsis mensurantur, ita describi posse deprehendentur. Sit in ellipsi proposita ratio amborum axium principalium $= 1 : m$, ac posito radio circuli mobilis $= r$, capiatur distantia puncti describentis ab ejus centro sive $= \frac{m+1}{m-1} r$, sive $= \frac{m-1}{m+1} r$. Tum si iste circulus super quocunque alio circulo sive extus sive intus provolvatur, ab utroque puncto describente semper ejusmodi curva describetur, cujus rectificatio cum rectificatione ellipsis propositae conveniet. Quo autem curvae hoc modo descriptae fiant algebraicae, necesse est, ut radius circuli mobilis ad radium circuli immoti rationem teneat rationalem, quae quo fuerit simplicior, eo minus curvae descriptae erunt compositae: ac constituto quidem circulo immoto, sive mobilis extra eum, sive intra volvatur, tum vero sive punctum describens extra sive intra circulum mobilem accipitur, quaternae illae curvae describentur, quas conjunctas inveneramus.

25. Operae pretium fore videtur harum linearum epi- et hypocycloidalium proprietates primarias, quatenus huc pertinent, ac praecipue earum rectificationem attentius contemplari. Sit igitur * (Fig. 50, 51 item 52, 53) C centrum circuli immoti AQ, ejusque radius $CA = CQ = a$, super cujus peripheria volvatur circulus $OLRQV$, cujus radius $OQ = OR = r$; sitque punctum describens M in radio OR, ac vocetur $OM = \mu r$, ita ut sit sive $\mu = \frac{m+1}{m-1}$, sive $\mu = \frac{m-1}{m+1}$. Hoc modo a stilo M descripta sit curva DM, cujus initium D ei respondeat circuli mobilis situi, quo punctum R tangebat circulum immotum in A. Hinc ergo ex natura motus volutorii erit arcus QR aequalis arcui QA. Quare si dicamus angulum $ACQ = \varphi$, ob arcum $AQ = QR = a\varphi$, erit angulus $QOR = \frac{a}{r}\varphi$. Vocemus autem brevitatis gratia hunc angulum $QOR = \alpha\varphi$, ut sit $\alpha = \frac{a}{r}$. Tum

vero ex punctis M et O ad rectam CA pro axe assumtam demittantur perpendicula MP et OS, itemque ex M in rectam MT axi AC parallelam, sintque coordinatae orthogonales curvae descriptae $CP = x$ et $PM = y$.

26. Cum jam sit angulus $ACQ = \varphi$ et $CO = a \pm r$, ubi signum superius pro curvis epicycloidalibus, inferius vero pro hypocycloidalibus valet, erit $CS = (a \pm r)\cos\varphi$ et $OS = (a \pm r)\sin\varphi$. Deinde ob ang. $COS = 90° - \varphi$ et $COR = \alpha\varphi$, erit ang. $MOT = (\alpha + 1)\varphi - 90°$ pro epicycloidalibus (Figg. 50, 51), at pro hypocycloidalibus (Figg. 52, 53), ob $COS = 90° - \varphi$ et $COR = 180° - \alpha\varphi$, * erit ang $MOT = 90° - (\alpha - 1)\varphi$, unde ex triangulo OMT ad T rectangulo, ob latus $OM = \mu r$, obtinebimus pro utroque casu

curvarum epicycloidalium Fig. 50 et 51:

$$MT = -\mu r \cos(\alpha + 1)\varphi,$$
$$OT = +\mu r \sin(\alpha + 1)\varphi,$$

ergo
$$CP = (a + r)\cos\varphi - \mu r\cos(\alpha + 1)\varphi = x,$$
$$PM = (a + r)\sin\varphi - \mu r\sin(\alpha + 1)\varphi = y.$$

curvarum hypocycloidalium Fig. 52 et 53:

$$MT = \mu r\cos(\alpha - 1)\varphi,$$
$$OT = \mu r\sin(\alpha - 1)\varphi,$$

ergo
$$CP = (a - r)\cos\varphi + \mu r\cos(\alpha - 1)\varphi = x,$$
$$PM = (a - r)\sin\varphi + \mu r\sin(\alpha - 1)\varphi = y.$$

Consequenter pro utroque casu conjunctim

$$CP = x = (a \pm r)\cos\varphi \mp \mu r\cos\left(1 \pm \frac{a}{r}\right)\varphi,$$

$$PM = y = (a \pm r)\sin\varphi \mp \mu r\sin\left(1 \pm \frac{a}{r}\right)\varphi,$$

27. Hinc ergo videmus totum discrimen inter has curvas epicycloidales et hypocycloidales tantum in signo quantitatis r esse situm, ita ut omnes his expressionibus pro coordinatis $CP = x$ et $PM = y$ possimus complecti

$$x = (a + r)\cos\varphi - \mu r\cos\left(1 + \frac{a}{r}\right)\varphi,$$

$$y = (a + r)\sin\varphi - \mu r\sin\left(1 + \frac{a}{r}\right)\varphi,$$

quae proprie ad epicycloidales pertinent, sed sumta quantitate r negativa simul ad hypocycloidales extenduntur. Differentiando ergo habebimus

$$dx = -(a + r)\,d\varphi\left(\sin\varphi - \mu\sin\left(1 + \frac{a}{r}\right)\varphi\right),$$

$$dy = +(a + r)\,d\varphi\left(\cos\varphi - \mu\cos\left(1 + \frac{a}{r}\right)\varphi\right),$$

unde elementum arcus hujus curvae $\sqrt{(dx^2 + dy^2)} = ds$ reperitur

$$ds = (a + r)\,d\varphi\sqrt{\left(1 + \mu\mu - 2\mu\cos\frac{a}{r}\varphi\right)},$$

et radius osculi in M ita erit expressus:

$$\frac{(a + r)\left(1 + \mu\mu - 2\mu\cos\frac{a}{r}\varphi\right)^{\frac{3}{2}}}{1 + \mu\mu - \mu\left(2 + \frac{a}{r}\right)\cos\frac{a}{r}\varphi}.$$

28. Quaecunque igitur hujusmodi curva descripta dabitur ellipsis, in qua arcui curvae DM arcus aequalis assignari poterit. Sit (Fig. 54) $adbe$ haec ellipsis, ejusque axes orthogonales ab et de; * vocetur semiaxis minor $ca = cb = c$ et semiaxis major $cd = ce = mc$, sumtaque super illo a centro c abscissa $cp = z$, erit applicata $pm = m\sqrt{(cc - zz)}$ et arcus ellipticus $dm = \int dz\sqrt{\left(1 + \frac{mmzz}{cc - zz}\right)}$. Statuatur $z = c\sin\theta$, eritque hic arcus

$$dm = \int c\,d\theta \sqrt{(1 + (mm - 1)\sin^2\theta)} = \int c\,d\theta \sqrt{(\tfrac{1}{2}(mm + 1) - \tfrac{1}{2}(mm - 1)\cos 2\theta)},$$

quae forma ut illi pro ds inventae aequalis reddatur, fieri oportet

$$\theta = \frac{a}{2r}\varphi = \frac{1}{2} QOR \quad \text{et} \quad \frac{mm+1}{mm-1} = \frac{1+\mu\mu}{2\mu}, \quad \text{seu} \quad m = \pm\frac{\mu+1}{\mu-1}$$

vel, quod eodem redit, capiatur $m = \frac{VM}{RM}$ in Figg. 50, 51, 52, 53, eritque arcus ellipticus

$$dm = \int \frac{acd\varphi}{2(\mu-1)r} \sqrt{(1 + \mu\mu - 2\mu\cos\frac{a}{r}\varphi)}.$$

Superest ergo, ut sit $\frac{ac}{2(\mu-1)r} = a + r$, unde semiaxes ellipsis fiunt

$$ca = cb = \frac{2(\mu-1)r(a+r)}{a} \quad \text{et} \quad cd = ce = \frac{2(\mu+1)r(a+r)}{a}.$$

29. In genere ergo habebimus hanc constructionem pro ellipsi quaesita:

$$\text{semiaxis } ca = \frac{2RM.CO}{CQ} \text{ et semiaxis } cd = ce = \frac{2VM.CO}{CQ},$$

qua descripta circa centrum C radio $ca = cb$ delineetur circulus $afbg$, tum ducatur radius cn ita, ut sit angulus $fen = \frac{1}{2} QOR$, et per n ducta recta pnm axi majori de parallela, erit arcus ellipticus dm aequalis arcui curvae supra descriptae DM. Unde patet, si circulus mobilis jam per semiperipheriam fuerit provolutus, quod evenit cum punctum V circulo immoto applicabitur, tum longitudinem curvae descriptae aequalem fore quadranti elliptico dma. Cum autem circulus mobilis integram revolutionem absolverit, tractus curvae descriptae semiperipheriae ellipticae dae erit aequalis; sicque uti ellipsis est curva in se rediens, ita provolutione continuata longitudo curvae continuo crescet.

30. De his curvis adhuc notari meretur ipsam quoque ellipsin inter eas comprehendi. Si enim pro hypocycloidalibus sumatur radius circuli immoti aequalis diametro circuli mobilis seu $a = 2r$, vel si in nostris formulis § 27 ponamus $r = -\frac{1}{2}a$, habebimus

$$x = \frac{1}{2}a\cos\varphi + \frac{1}{2}\mu a\cos\varphi = -(1+\mu)r\cos\varphi,$$

$$y = \frac{1}{2}a\sin\varphi - \frac{1}{2}\mu a\sin\varphi = -(1-\mu)r\sin\varphi,$$

unde prodit

$$\frac{xx}{(1+\mu)^2} + \frac{yy}{(1-\mu)^2} = rr,$$

quae est aequatio pro ellipsi, cujus semiaxes sunt $(\mu-1)r$ et $(\mu+1)r$ seu MR et MV, estque ea ipsa ellipsis, cujus arcubus nostrae curvae mensurantur: nam ob $CQ = 2CO$ fit utique $ca = RM$ et $cd = VM$. Potest itaque quaecunque ellipsis provolutione circuli intra peripheriam alterius circuli, cujus radius duplo est major, describi, ubicunque enim tum stylus in circulo mobili figatur, ab eo ellipsis describetur.

31. Innumerabiles autem curvae, quae sint cum arcubus parabolicis commensurabiles, quarum supra unam exhibui, seu ut positis coordinatis x et y, sit

$$\sqrt{(dx^2 + dy^2)} = dz\sqrt{(1 + zz)},$$

sequenti modo se habebunt. Ponatur $z = \frac{2}{n}$ tang φ seu tang $\varphi = \frac{1}{2} nz$, ac statuatur

$$x = \frac{2 \sin n\varphi}{nn \cos^2 \varphi} \quad \text{et} \quad y = \frac{2 \cos n\varphi}{nn \cos^2 \varphi},$$

erit semper, quicunque numerus pro n assumatur, $\int \sqrt{(dx^2 + dy^2)} = \int dz \sqrt{(1 + zz)}$. Facile autem ang. φ eliminatur ob $\sqrt{(xx + yy)} = \frac{2}{nn \cos^2 \varphi}$, unde fit

$$\cos \varphi = \frac{\sqrt{2}}{n \sqrt[4]{(xx + yy)}} \quad \text{hincque} \quad \frac{y}{\sqrt{(xx + yy)}} = \cos n\varphi.$$

At si variabilem z retinere velimus, erit

$$x = \frac{\frac{n}{1} \cdot \frac{nz}{2} - \frac{n(n-1)(n-2)}{1.2.3} \cdot \frac{n^3 z^3}{8} + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} \cdot \frac{n^5 z^5}{32} - \text{etc.}}{\frac{1}{2} nn \left(1 + \frac{nnzz}{4}\right)^{\frac{n-2}{2}}},$$

$$y = \frac{1 - \frac{n(n-1)}{1.2} \cdot \frac{nnzz}{4} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} \cdot \frac{n^4 z^4}{16} - \text{etc.}}{\frac{1}{2} nn \left(1 + \frac{nnzz}{4}\right)^{\frac{n-2}{2}}},$$

quae formulae, quoties n sumitur numerus integer positivus, finito terminorum numero constabunt. Verum priores semper, etiamsi pro n statuatur numerus fractus, ad aequationem finitam deducunt. Veluti si $n = \frac{1}{2}$, cum sit

$$\cos \varphi = \frac{2\sqrt{2}}{\sqrt[4]{(xx + yy)}} \quad \text{et} \quad \cos \frac{1}{2} \varphi = \frac{y}{\sqrt{(xx + yy)}},$$

erit hinc $\cos \varphi = \frac{2yy}{xx + yy} - 1 = \frac{yy - xx}{xx + yy}$, unde obtinetur

$$\frac{64}{xx + yy} = \frac{(yy - xx)^4}{(xx + yy)^4} \quad \text{seu} \quad (yy - xx)^4 = 64 (xx + yy)^3$$

pro linea ordinis octavi.

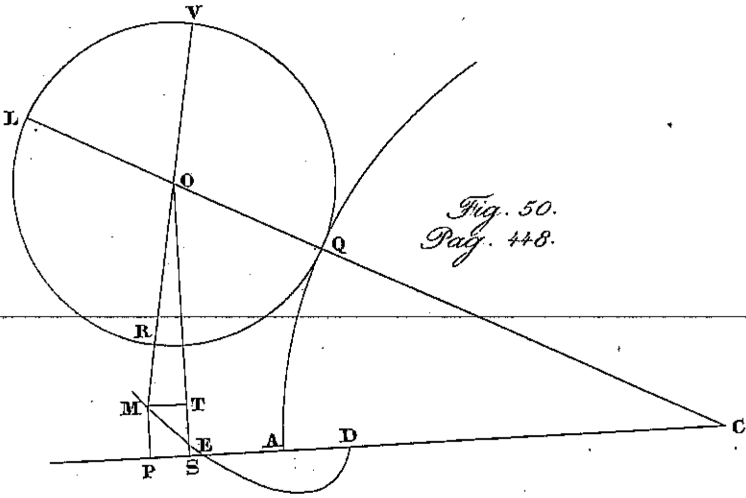

Fig. 50.
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Fig. 54.
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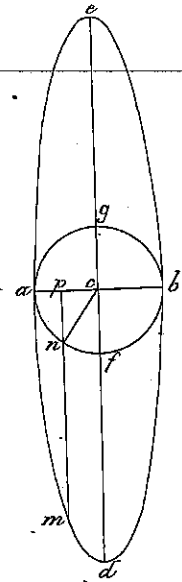

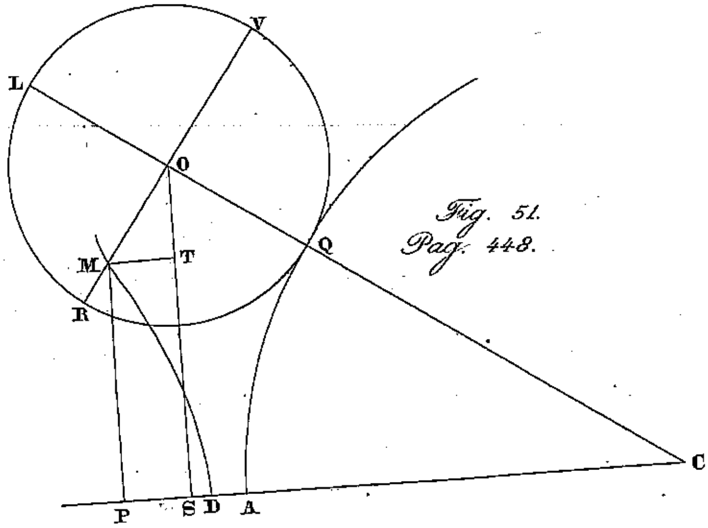

Fig. 51.
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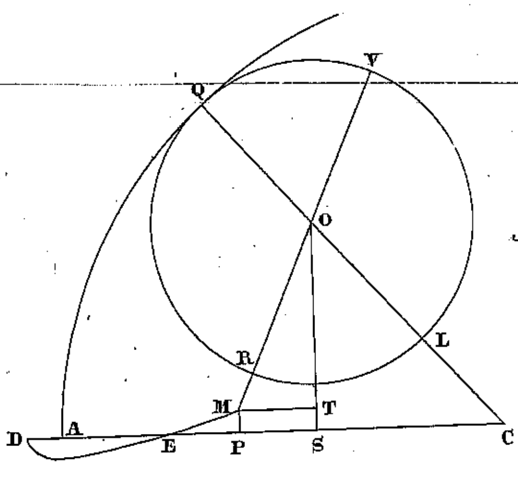

Fig. 52.
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Fig. 53.
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